Inequality in triangle involving medians

https://www.linkedin.com/groups/8313943/8313943-6369046791299624961

Let m_a, m_b, m_c be lengths of the medians of a triangle ABC.Prove that

$\dfrac{9}{4R+r} \leq \dfrac{1}{m_a} + \dfrac{1}{m_b} + \dfrac{1}{m_c} \leq \dfrac{1}{r}$.

Solution by Arkady Alt , San Jose, California, USA.

Let h_a, h_b, h_c be lenghts of heighs of a triangle ABC and F be it's area

Since $m_x \geq h_x, xh_x = F, x \in \{a, b, c\}$ and $F = sr$,where s is semiperimeter,

then $\dfrac{1}{m_a} + \dfrac{1}{m_a} + \dfrac{1}{m_a} \leq \dfrac{1}{h_a} + \dfrac{1}{h_b} + \dfrac{1}{h_c} = \dfrac{a}{2F} + \dfrac{b}{2F} + \dfrac{c}{2F} = \dfrac{s}{F} = \dfrac{1}{r}$.

Since by Cauchy Inequality $\dfrac{1}{m_a} + \dfrac{1}{m_b} + \dfrac{1}{m_c} \geq \dfrac{9}{m_a + m_b + m_c}$ remains to prove inequality

(1) $m_a + m_b + m_c \leq 4R + r$.

Noting that $(m_a + m_b + m_c)^2 = \dfrac{3\left(a^2 + b^2 + c^2\right)}{4} + 2\sum_{cyc} m_a m_b$,

$$m_a m_b \leq \frac{2c^2 + ab}{4} \qquad [1]$$

and $a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right), ab + bc + ca = s^2 + 4Rr + r^2$,

$s^2 \leq 4R^2 + 4Rr + 3r^2$(Gerretsen's Inequality) and $R \geq 2r$ we obtain

$(m_a + m_b + m_c)^2 \leq \dfrac{7\left(a^2 + b^2 + c^2\right) + 2(ab + bc + ca)}{4} = \dfrac{7 \cdot 2\left(s^2 - r^2 - 4Rr\right) + 2\left(s^2 + 4Rr + r^2\right)}{4} =$

$\dfrac{16s^2 - 12r^2 - 48Rr}{4} = 4s^2 - 3r^2 - 12Rr \leq$

$4\left(4R^2 + 4Rr + 3r^2\right) - 3r^2 - 12Rr =$

$16R^2 + 4Rr + 9r^2 = 16R^2 + 8Rr + r^2 - \left(4Rr - 8r^2\right) =$

$(4R + r)^2 - 4r(R - 2r) \leq (4R + r)^2$.

1. Sidelengths majorant for product of two medians–**problem 5291, SSMA February 2014,**

two solutions in May issue 2014,p.6,7

http://ssma.play-cello.com/wp-content/uploads/2016/03/May-2014-2.pdf,

or

http://www.equationroom.com/Publications/Suggested%20problems%20(with%20solutions)/
In%20School%20Science%20and%20Mathematics%20Journal%20(SSMJ)/5291-
Solutions%20May-2014.pdf

and two my solutions

http://www.equationroom.com/Publications/Suggested%20problems%20(with%20solutions)/
In%20School%20Science%20and%20Mathematics%20Journal%20(SSMJ)/5291_2%20my%20solutions.pdf